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A PROBLEM OF DYER, PORCELLI, AND ROSENFELD

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ABSTRACT

In this paper, we will demonstrate that a conjecture of Dyer, Porcelli, and Rosenfeld is correct. In fact, we will show that if M is any finite von Neumann factor and A is any non-zero element of M, then there exists a T in M such that the spectrum of $T+A$ is disjoint from the spectrum of T, i.e. such that $\sigma(T+A)\cap\sigma(T)=\varnothing$.

1. Introduction

In this paper, the author presents a solution to a problem posed in [2] by Dyer, Porcelli, and Rosenfeld for a II_1 factor. An entirely different solution has been obtained independently by Stratilia and Zsido [5]. Their solution extends the argument given in [2] for a I_n factor. We extend the argument given in [2] for a I_∞ factor. The crux of our argument is to extend lemma 4 of [2] which states that for all Hermitian elements A not belonging to a proper two sided ideal of a I_∞ factor ($A = A^*$), there exists a Hermitian T in the factor such that the spectrum of $T+iA$ contains no real numbers. ($i=\sqrt{-1}$.) We will extend this result to the following theorem:

THEOREM B. *Let A be a non-zero Hermitian element of a finite factor M. There exists a Hermitian element T of M such that the spectrum of $T+iA$ contains no real numbers, i.e. such that*

$$\sigma(T+iA)\cap\boldsymbol{R}=\varnothing.$$

These results have previously appeared in the author's dissertation [1], where it is also shown that Theorems B and C generalize to any W^*-algebra but that Theorem C does not generalize to the shift algebra, the C^* algebra generated by the Toeplitz operators with continuous symbol.

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2. The matrices P_n and T_n

In this section we prove a very special case of Theorem B. Let T_n be an $n \times n$ matrix with ones on the subdiagonal and superdiagonal and zeroes everywhere else. Let P_n be a matrix with a one in the lower right hand corner and zeroes everywhere else. For $n = 1$, we take $T_1 = 0$ and $P_1 = 1$. For $n = 3$, the pictures are

$$T_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and

$$P_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

LEMMA 2.1. $\sigma(T_n + iP_n) \cap \boldsymbol{R} = \varnothing$.

PROOF. Let $p_n(t)$ be the determinant of $T_n + iP_n - t$, and let $q_n(t)$ be the determinant of $T_n - t$. Hence $p_1(t) = i - t$ and $q_1(t) = -t$. Since the determinant is linear in the last row, $p_n(t) = q_n(t) + iq_{n-1}(t)$, if we take $q_0(t)$ to be identically equal to one. On the other hand,

$$q_n(t) = -tq_{n-1}(t) - q_{n-2}(t)$$

with initialization $q_0(t) = 1$ and $q_1(t) = -t$.

Now suppose $t \in \sigma(T_n + iP_n) \cap \boldsymbol{R}$. Then $p_n(t) = 0$ and hence $q_n(t) = q_{n-1}(t) = 0$. Hence $q_0(t) = 0$ by the above recursion relation, but this contradicts $q_0(t) = 1$. Hence $\sigma(T_n + iP_n) \cap \boldsymbol{R} = \varnothing$. Q.E.D.

3. The case when M is a I_n factor

In this section we will prove Theorem B when M is a I_n factor. In this case, M is isomorphic to the algebra of linear transformations on a finite dimensional Hilbert space H. Let $0 \neq A = A^* \in M$. We will use P_n and T_n of the previous section to construct a $T = T^* \in M$ such that $\sigma(T + iA) \cap \boldsymbol{R} = \varnothing$.

The following construction will yield $T = 0$ if A is invertible. In this case, $\sigma(T + iA) \cap \boldsymbol{R} = \sigma(iA) \cap \boldsymbol{R} = \varnothing$.

Choose an orthonormal basis $\{e_i\}_{i=1}^k$ for H so that A is represented by a diagonal matrix. Reorder the basis so that the first m basis vectors span the kernel of A $(m < k)$. Let $n = m + 1$. Let $a \in \sigma(A)$ be the first non-zero entry on the diagonal of the matrix of A. The matrix of A is either equal to aP_n or is equal to the direct sum of aP_n with an invertible matrix $D = D^*$. We may write $A = aP_n \oplus D$ with the understanding that D may be void. Let $T = aT_n \oplus 0$ if $D \neq \varnothing$ and $T = aT_n$ if $D = \varnothing$. Then $\sigma(iD) \cap R = \varnothing$ and

$$\sigma(T + iA) = \sigma(aT_n + iaP_n) \cup \sigma(iD)$$

so that $\sigma(T + iA) \cap \boldsymbol{R} = \varnothing$ by Lemma 2.1. Q.E.D.

In the above proof, we have used the fact that the spectrum of a direct sum of matrices is the union of the spectrum of the matrices. In general we will say that Q is the direct sum of E and F, $Q = E \oplus F$, if $Q = E + F$ and there exists a projection S $(S = S^2 = S^*)$ such that $E = SES$ and $F = (1 - S)F(1 - S)$. We will use the facts that the inverse of a direct sum is the direct sum of the inverses and the norm of a direct sum is the supremum of the norms without comment in the following sections.

The proof of Theorem B when M is a II_1 factor is similar to the above argument. The diagonalization of A is replaced by the spectral decomposition of A. The orthonormal basis e_i for H is replaced by a set of equivalent orthogonal projections $P_{h,i}$ in M. The major difficulty is that the equality $A = aP_n \oplus D$ is replaced by an approximation $A \sim \Sigma_i^{\oplus} a_i P_{n,i} \oplus D$. The development of these ideas is the subject of the next two sections.

4. A technical lemma

M will denote a II_1 factor throughout this section. A will be a non-zero Hermitian element of M with zero and one in its spectrum. In this section, we will prove a technical lemma which will allow A to be approximated by a linear combination of projections in a very special manner. These projections will be closely related to the spectral family $\{P(E)\}_{E \subset \boldsymbol{R}}$ of A (E runs over the Borel subsets of $\boldsymbol{R}$, and $A = \int_{\boldsymbol{R}} x dP(x)$.)

For any projection $X \in M$, dim X will denote the von Neumann dimension of X. Since $1 \in \sigma(A)$, we may choose a positive number c such that

$$0 < c < \dim P[3/4, 5/4). \tag{1}$$

In other words, c is less than the dimension of the spectral projection of A over the half open interval [3/4, 5/4).

LEMMA 4.1. *For any positive integer k, there exists k numbers,* $3/4 \leqq a_1 \leqq \cdots \leqq a_k \leqq 5/4$, *and k orthogonal projections,* P_i $(i = 1, \cdots, k)$, *such that*

$$P_i \text{ commutes with } A, \tag{2}$$

$$\dim P_i = c/k, \tag{3}$$

and

$$P(a_{i-1}, a_i) \leqq P_i \leqq P[a_{i-1}, a_i] \text{ for all } i \in \{1, \cdots, k\} \tag{4}$$

where $a_0 = 3/4$.

PROOF. The proof is by induction on i. As our induction hypothesis we take the conclusions of the lemma and

$$P[a_0, a_i) \leqq \sum_{j=1}^{i} P_j \leqq P[a_0, a_i]. \tag{5}$$

For $i = 1$, we define

$$a_1 = \sup \{ d \in \boldsymbol{R} \mid \dim P[a_0, d) < c/k \}$$

where we take $P[a_0, d) = 0$ if $d \leqq a_0$. Hence $a_0 \leqq a_1$. Condition (1) insures that $a_1 \leqq 5/4$. The continuity of the von Neumann dimension function and the fact that $P[a_0, a_1)$ is the supremum of $\{P[a_0, d) \mid d < a_1\}$ imply that $\dim P[a_0, a_1) \leqq c/k$. Similarly, $\dim P[a_0, a_1] \geqq c/k$. Hence $\dim P[a_1, a_1] \geqq c/k - \dim P[a_0, a_1) \geqq 0$ and we may choose a projection $Q \leqq P[a_1, a_1]$ such that $\dim Q = c/k - \dim P[a_0, a_1)$.

If we let $P_1 = P[a_0, a_1) \oplus Q$, we will have satisfied our induction hypothesis for $i = 1$. In fact P_1 commutes with A since it commutes with the spectral family of A so that (2) follows. The calculation $\dim P_1 = \dim P[a_0, a_1) + c/k - \dim P[a_0, a_1) = c/k$ demonstrates (3). P_1 satisfies (4) and (5) since $Q \leqq P[a_1, a_1]$. Finally, $a_1 < 5/4$ since $\dim P[a_0, a_1) \leqq c/k \leqq c < \dim P[3/4, 5/4)$ by (5) and (1).

Fix i such that $1 < i \leqq k$. Suppose that a_j and P_j have been constructed to satisfy our induction hypothesis for all $j < i$.

Let $X = 1 - \Sigma_{j=1}^{i-1} P_j$. We will mimic the argument for the case $i = 1$ by replacing a_1 with a_i and $P(E)$ with $P(E)X$. Hence we define

$$a_i = \sup \{ d \in \boldsymbol{R} \mid \dim P[a_0, d)X < c/k \}.$$

Condition (5) with i replaced by $i - 1$ implies that $P[a_0, d)X = P[a_{i-1}, d)X$. Hence $a_{i-1} \leqq a_i$. As before, $\dim P[a_0, a_i)X \leqq c/k$ and $\dim P[a_0, a_i]X \geqq c/k$.

Hence we may choose a projection $Q_i \leqq P[a_i, a_i]X$ such that $\dim Q_i = c/k - \dim P[a_0, a_1)X$. If we let $P_i = P[a_0, a_i)X \oplus Q_i$, then we will satisfy our induction hypothesis. In fact, (2) and (3) follow precisely as in the case when $i = 1$. (5) may be demonstrated as follows:

$$P[a_0, a_i) = P[a_0, a_i) \sum_{j=1}^{i-1} P_j \oplus P[a_0, a_i) X \leqq \sum_{j=1}^{i} P_i \leqq P[a_0, a_i].$$

(4) is a consequence of (5), and the fact that $a_i \leqq 5/4$ follows from (1) and (5):

$$\dim P[a_0, a_i) \leqq ci/k \leqq c < \dim P[3/4, 5/4). \qquad \text{Q.E.D.}$$

5. The case when M is a II_1 factor

In this section, we prove Theorem B when M is a II_1 factor. The proof will be an application of the matrices T_n and P_n as in the case when M is a I_n factor. The basic constant governing the use of T_n and P_n is the supremum of the norms of $(T_n + iP_n - t)^{-1}$,

$$d_n = \sup_t \| (T_n + iP_n - t)^{-1} \|,$$

where $t \in \mathbf{R}$. $d_n < \infty$ by Lemma 2.1. The reason d_n is crucial is the following lemma.

LEMMA 5.1. *Let B and C be elements of a Banach algebra with identity. Suppose that $\sigma(B) \cap \mathbf{R} = \emptyset$ and that*

$$\| C - B \| < 1/ \sup_t \| (B - t)^{-1} \|$$

where t varies over the real numbers $\mathbf{R}$. Then

$$\sigma(C) \cap \mathbf{R} = \emptyset.$$

The proof of Lemma 5.1 follows from the equality

$$(C - t)^{-1} = (B - t)^{-1} \sum_{n=0}^{\infty} [(B - C)(B - t)^{-1}]^n$$

which holds for $\| (B - C) \| \, \| (B - t)^{-1} \| < 1$ (see Halmos [3], pp. 53, 151, 245, 248). With these preliminaries disposed of, we are ready to prove Theorem B for a II_1 factor M.

Let $0 \neq A = A^* \in M$. Without loss of generality, we may assume that 0 and 1 are in the spectrum of A. $P(E)$ will be the spectral family of A and "dim" will be the von Neumann dimension function as in the previous section.

The construction of a T such that $\sigma(T + iA) \cap \mathbf{R} = \emptyset$ and $T = T^*$ proceeds in six steps:

1) Let m be a positive integer such that

$$(\dim P[-1/4,\ 1/4])/m < (\dim P[3/4,\ 5/4))/2.$$

2) Let $n = m + 1$. Let $\varepsilon > 0$, $\varepsilon < 1/4$, and $\varepsilon < (3/4)/d_n$.

3) Choose a positive integer $k > 1/(2\varepsilon)$ and divide $P[-\varepsilon,\ \varepsilon]$ into mk equivalent orthogonal projections $P_{h,i}$ ($h = 1, \cdots, m$ and $i = 1, \cdots, k$):

$$P[-\varepsilon,\ \varepsilon] = \sum_{h,i}^{\oplus} P_{h,i}.$$

From step 1), $\dim P_{1,1} = \dim P[-\varepsilon,\ \varepsilon]/mk < \dim P[3/4,\ 5/4)/2k$. Loosely speaking, this means that $2k$ copies of $P_{1,1}$ will fit into $P[3/4,\ 5/4)$.

4) We may apply Lemma 4.1 with c replaced by $2 \dim P[-\varepsilon,\ \varepsilon]/m$ and k replaced by $2k$ to obtain $2k$ numbers a_i and $2k$ orthogonal projections P'_i which are equivalent to $P_{1,1}$ and which satisfy

i) P'_i commutes with A,

ii) $3/4 = a_0 \leqq a_1 \cdots \leqq a_{2k} \leqq 5/4$,

iii) $P(a_{i-1},\ a_i) \leqq P'_i \leqq P[a_{i-1},\ a_i]$ for all $i = 1, \cdots, 2k$.

5) Choose k of the ordered pairs $(P'_i,\ a_i)$ with $a_i - a_{i-1} < \varepsilon$ and call them $(P_{n,i},\ a_i)$. Now $i = 1, \cdots, k$ again. k such P'_i exist for otherwise

$$\sum_{i=1}^{2k} a_i - a_{i-1} > k\varepsilon > 1/2$$

by our choice of k. But this contradicts condition ii) on the a_i. Recall that we set $n = m + 1$ in step two.

The point is that

$$\left\| A \sum_{h,i}^{\oplus} P_{h,i} - \sum_{i}^{\oplus} a_i P_{n,i} \right\| \leqq \varepsilon$$

where $h = 1, \cdots, n$ and $i = 1, \cdots, k$, as follows from the spectral theorem, the fact that $a_i - a_{i-1} < \varepsilon$, and the choice of the $P_{h,i}$ in steps 3) and 4)†. Furthermore, $A(1 - \Sigma^{\oplus}_{h,i} P_{h,i})$ is invertible in $(1 - \Sigma^{\oplus}_{h,i} P_{h,0}) M (1 - \Sigma^{\oplus}_{h,i} P_{h,i})$.

6) Let $T_{n,i}$ have the matrix of T_n in the basis $P_{1,i}, \cdots, P_{n,i}$. In other words, let $T_{n,i} = \Sigma^{m}_{h=1} U_{h,i} + U^*_{h,i}$ where $U_{h,i}$ is a partial isometry with initial domain $P_{h,i}$ and final range $P_{h+1,i}$. Let $T = \Sigma^{\oplus}_i a_i T_{n,i}$. Now apply Lemma 5.1 in the algebra $\Sigma^{\oplus}_{h,i} P_{h,i} M \Sigma^{\oplus}_{h,i} P_{h,i}$ with $B = T + i\Sigma^{\oplus}_i a_i P_{n,i}$ and $C = T + iA \Sigma_{h,i} P_{h,i}$ to obtain

† $\| A \Sigma^{\oplus}_{h,i} P_{h,i} - \Sigma^{\oplus}_i a_i P_{n,i} \| = \sup\{ \| A \Sigma^{\oplus m}_{h=1,}{}^{k}_{i=1} P_{h,i} \|,\ \| A \Sigma^{\oplus}_i P_{n,i} - \Sigma^{\oplus}_i a_i P_{n,i} \| \} \leqq \sup_i \{ \| AP(-\varepsilon, \varepsilon) \|,$ $\| (A - a_i) P[a_{i-1}, a_i] \| \} \leqq \varepsilon$.

$$\sigma(T + iA) \cap \boldsymbol{R} = \varnothing .$$

Lemma 5.1 applies since $\sigma(B) \cap \boldsymbol{R} = \varnothing$ by Lemma 2.1† and

$$\| B - C \| = \left\| A \sum_{h,i}^{\oplus} P_{h,i} - \sum_{i}^{\oplus} a_i P_{n,i} \right\| \leqq \varepsilon < (3/4)/d_n$$

and

$$\sup \left\| \left(T + i \sum_i a_i P_{n,i} - t \right)^{-1} \right\| < d_n/(3/4)$$

since $1/a_i < 1/(3/4)$. Q.E.D

6. Conclusion

Section 5 completes the proof of Theorem B. Using the same argument as in the proof of Theorem B of [2], we obtain our Theorem C:

THEOREM C. *For all non-zero A in a finite factor M, there exists a $T \in M$ such that $\sigma(T + A) \cap \sigma(T) = \varnothing$.*

It is easy to see that if A is contained in a proper two sided ideal of a Banach algebra with identity, M, then $\sigma(T + A) \cap \sigma(T) \neq \varnothing$ for all $T \in M$ (this is theorem A of [1]). Hence the well-known fact that the finite factors are simple (i.e. they have no non-zero proper two sided ideals) is a consequence of our arguments. It would be interesting to know if Theorem C is true when M is an arbitrary simple C^*-algebra.

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† In order to obtain the last inequality in Section 5 and to check the applicability of Lemma 2.1, one needs the fact that the norm of a matrix equals the norm of an operator with that matrix. This fact may be obtained directly from the definition $\| A \| = \sup \{ | A(x) | \mid | x | = 1 \}$ or it may be derived as a consequence of the theorem that a C^*-algebra *-isomorphism is norm preserving [6, p. 5]. The latter method may be exposed as follows. Let $Q_1, \cdots, Q_n$ be n equivalent orthogonal projections in a C^*-algebra M. Let $V_{i,j}$ be a set of partial isometries such that 1) $V_{i,j}$ has initial domain Q_j and final range Q_i, 2) $V_{i,k} V_{k,j} = V_{i,j}$ and 3) $V^*_{i,k} = V_{k,i}$ for all $i, k = 1, \cdots, n$. The map Φ from the n by n matrices into M defined by $\Phi(a_{i,j}) = \Sigma a_{i,j} V_{i,j}$ is a *-isomorphism. Hence the norm, the norm of the inverse, and the spectrum of the operator $\Phi(a_{i,j})$ are the same as the norm, the norm of the inverse, and the spectrum of its matrix $(a_{i,j})$.

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